Transient Catalytic Ignition on a Flat Plate with External Energy Flux

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This paper deals with the transient analysis of the catalytic ignition of premixed gases over a flat plate of finite thickness. The ignition process is facilitated by an external heat flux added to the combustible gases through the plate. Activation energy asymptotic methods based on high Zel'dovich numbers are used to deduce analytical expressions for the critical time for ignition. This ignition process can be characterized by three different stages: inert heating, transition, and diffusion-controlled completion. The influence of the finite thermal conductivity of the plate, which permits feedback of the heat on the transient ignition process, is also evaluated.

Introduction

EXPERIMENTAL and theoretical investigations of catalytic combustion have received special attention in the literature in the past. This combustion process represents an important alternative to conventional burners, especially in gas turbine combustion chambers. With the use of catalytic combustion, the nitric oxides emissions can be greatly reduced. There are two types of catalytic combustion, depending on the existence of the homogeneous reaction. In both types, however, it is necessary to ignite the catalytic reaction.

Catalytic ignition in a flat-plate boundary layer has been the focus of several studies. Using the local similarity concept, Artyuk et al.[1] numerically solved governing equations for an adiabatic plate. Lindberg and Schmitz[2,3] studied numerically the surface ignition process for a plate and wedge type of boundary-layer flow. These analyses were made for the limiting cases of adiabatic and perfect conducting plates. Mihail and Teodorescu[4] used a refined numerical analysis to solve the integral governing equations by applying the Lighthill approximation[5] for high Prandtl and Schmidt numbers. Ahluwalia and Chung[6] analyzed the same problem, but they solved the integral governing equations through an erroneous utilization of the Laplace method. For an adiabatic plate, these studies show the transition from a kinetically controlled process close to the leading edge to a diffusion-controlled process downstream. Liñán[7] showed that this transition occurs abruptly at a well-defined distance for large Zel'dovich numbers—i.e., for large values of the ratio of the activation energy of the global catalytic reaction to the thermal energy. Using activation energy asymptotic methods, Liñán and Treviño[8] studied the ignition and extinction of the catalytic reaction in the flow of a reacting mixture over a flat plate, including the longitudinal heat transfer through the plate due to finite values of the thermal conductivity. It was found that the critical Damköhler number for ignition is not strongly affected by this axial heat conduction. However, the finite thermal conductivity has a strong influence on the extinction process. In general, the ratio of the Damköhler number for ignition to that for extinction is very large for high Zel'dovich numbers. Thus, it is convenient to work with surface Damköhler numbers well below those for ignition. In order to ignit the catalytic reaction, it is necessary to have some external heat sources. In a recent paper, Treviño and Liñán[9] analyzed the steady-state catalytic ignition process generated by an external energy flux. The influence of the axial heat feedback due to the finite thermal conductivity of the plate was evaluated. They obtained the critical external heat flux for ignition as a function of the flow and physicochemical parameters. It was found that from a practical point of view, the total heat flux for ignition decreases as the Damköhler number decreases in the region of interest. Recently, Fakheri and Buckius[10] numerically solved the transient gas-phase governing equations to study catalytic ignition on a flat plate. They studied the two limiting cases of a perfect conducting plate and an adiabatic plate.

The objective of the present work is to study analytically the transient catalytic ignition process for flat-plate boundary-layer flow with external heat flux. The gas phase is assumed to be quasisteady. That is, the characteristic warm-up time for the plate is very large compared with the residence time in the gas phase. The influence of the axial heat feedback mechanism through the catalytic device is to be evaluated in the two limiting cases of a perfect conducting plate and an adiabatic plate.

Formulation

The physical model analyzed is one of a gaseous combustible mixture flowing parallel to a catalytic plate of finite thickness and thermal conductivity. The catalytic plate is externally heated in order to achieve the catalytic ignition. Using the Lighthill approximation[5] for high Prandtl (Pr) and Schmidt (Sc) numbers (which, in fact, gives good results for Pr and Sc of order unity) and assuming no chemical reactions and quasisteady behavior in the gas phase, the integrodifferential governing equations are given by

$$-k_0 C^n \exp\left(-\frac{T_a}{T}\right) = 0.332 D\sqrt{\frac{u_\infty}{\nu x}} Sc^{1/3} \int_{C_\infty}^{C} K(\tilde{x}, x) \mathrm{d}\tilde{C} \qquad (1)$$

Received April 3, 1984; revision received Nov. 5, 1984.

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$$-\rho_p c_p d\frac{\partial T}{\partial t}+\lambda_p d\frac{\partial^2 T}{\partial x^2}+(-\Delta H)k_0 C^n \exp\left(-\frac{T_a}{T}\right)+q_e$$

$$=0.332\lambda\sqrt{\frac{u_\infty}{\nu x}}Pr^{1/3}\left[\int_{T_\ell}^{T}K(\bar{x},x)\mathrm{d}T+T_\ell-T_\infty\right] \qquad (2)$$

with the kernel $K(\bar{x},x)$ given as

$$K(\bar{x},x)=\left[1-\left(\frac{\bar{x}}{x}\right)^{3/4}\right]^{-1/3}$$

In the above equations, an Arrhenious-type one-step catalytic reaction and a uniform plate temperature in the transverse direction, are assumed. Also, k_0 represents the preexponential term of the catalytic reaction; n the reaction order; Ta the activation temperature of the reaction; T and C the temperature and the reactant concentration on the surface of the plate, respectively; $(-\Delta H)$ the heat release per unit mole of fuel consumed; λ and λ_p the thermal conductivities of the gas and the plate, respectively; D the mass diffusion cofficients; ν the kinematic coefficient of viscosity; d the thickness of the plate; u_∞, T_∞, and C_∞ the velocity, temperature, and reactant concentration of the freestream, respectively; T_ℓ the temperature of the plate at the leading edge; x the longitudinal coordinate with the origin at the leading edge; ρ_p and c_p the density and the specific heat of the plate, respectively; and $q_e(x,t)$ the external heat flux per unit surface.

The initial and boundary conditions associated with Eqs. (1) and (2) for the adiabatic leading and trailing edges are given by $T(x,0)=T_0(x)$ and

$$\frac{\partial T}{\partial x}=0 \text{ at } x=0 \text{ and } x=L \text{ for all } t \qquad (3)$$

where L corresponds to the length of the plate.

Introducing the following nondimensional variables and coordinates:

$$\theta=\frac{T-T_\infty}{T_\infty\beta_1} \text{ with } \beta_1=\frac{(-\Delta H)DC_\infty Sc^{1/3}}{T_\infty\lambda Pr^{1/3}}$$

$$Y=\frac{C}{C_\infty}, \quad \tau=\frac{0.332\lambda\sqrt{u_\infty}Pr^{1/3}}{\rho_p c_p d\sqrt{\nu L}}t, \quad \chi=\frac{x}{L} \qquad (4)$$

the governing equations reduce to

$$-\frac{\delta}{\Gamma}Y^n\exp\left(\frac{\Gamma\theta}{1+\beta_1\theta}\right)=\frac{1}{\sqrt{\chi}}\int_1^Y K(\bar{\chi},\chi)\mathrm{d}\bar{Y} \qquad (5)$$

$$-\frac{\partial\theta}{\partial\tau}+q(\chi,\tau)+\alpha\frac{\partial^2\theta}{\partial\chi^2}+\frac{\delta}{\Gamma}Y^n\exp\left(\frac{\Gamma\theta}{1+\beta_1\theta}\right)$$

$$=\frac{1}{\sqrt{\chi}}\left[\int_{\theta_\ell}^{\theta}K(\bar{\chi},\chi)\mathrm{d}\bar{\theta}+\theta_\ell\right] \qquad (6)$$

where the nondimensional parameters are defined as

$$\delta=\frac{k_0 C_\infty^{n-1}\sqrt{\nu L}\Gamma\exp(-Ta/T_\infty)}{0.332\mathrm{D}\sqrt{u_\infty}Sc^{1/3}}$$

$$\Gamma=Ta\beta_1/T_\infty$$

$$\alpha=\frac{\lambda_p d\sqrt{\nu}}{0.332Pr^{1/3}\sqrt{u_\infty}L^{3/2}\lambda} \qquad (7)$$

$$q=\frac{q_e\sqrt{\nu L}}{0.332Pr^{1/3}\sqrt{u_\infty}\lambda T\beta_1}$$

The nondimensional initial and boundary conditions are then reduced to

$$\theta(\chi,0)=\theta_0(\chi)$$

$$\frac{\partial\theta}{\partial\chi}=0 \text{ at } \chi=0 \text{ and } \chi=1 \text{ for all} \qquad (8)$$

The parameter β_1 gives the ratio between the energy released by the chemical reaction to the thermal energy of the freestream and, in general, is of order unity. Γ is the Zeľdovich number defined as the ratio between the activation energy of the catalytic reaction to the thermal energy of the mixture; in combustion it is a large number. Also, δ is the Damköhler number and represents the ratio between the residence time to the reaction time; α the ratio of the ability of the plate to carry heat in the streamwise direction to the ability of the gas to carry heat from the plate, and q the ratio between the external heat flux added to the heat released by the catalytic chemical reaction.

Analysis

This section is an analysis of the transient ignition of the catalytic surface reaction on a flat plate with uniform external heat flux. At time $\tau=0$, the temperature of the plate is assumed to be identical to that of the freestream. At time $\tau>0$, a uniform heat flux will heat the plate. The temperature of the plate will increase in a way that depends strongly on the longitudinal heat flux (that is, on parameter α), reaching a time where the chemical reaction is important and leading finally to ignition. As shown in Ref. 9, it is convenient to work with Damköhler numbers δ well below to those of ignition, that is, with values of $\delta \ll 1$. In this case, it is necessary to add a nondimensional external heat flux q of order unity. The two limiting cases of a perfect conducting plate ($\alpha\to\infty$) and the adiabatic plate ($\alpha=0$) are analyzed in this section.

Perfect Conducting Plate ($\alpha\to\infty$)

In this limiting case, the temperature of the plate does not depend on the longitudinal coordinate. Thus, Eq. (6) can be integrated for the whole length of the plate, leading to

$$-\frac{\mathrm{d}\theta}{\mathrm{d}\tau}+q+\frac{\delta}{\Gamma}\exp\left|\frac{\Gamma\theta}{1+\beta_1\theta}\right|\int_0^1 Y^n\mathrm{d}\chi=2\theta \qquad (9)$$

In Eq. (9), both adiabatic edges of the plate were considered. In this case, the reactant concentration balance equation (5) can be reduced after integration to the form

$$-Y^n=\frac{1}{\sqrt{\xi}}\int_1^Y K(\bar{\xi},\xi)\mathrm{d}\bar{Y} \qquad (10)$$

where

$$\xi=\chi\left|\frac{\delta}{\Gamma}\exp\left(\frac{\Gamma\theta}{1+\beta_1\theta}\right)\right|^2$$

The energy balance equation (9) can then be transformed to

$$-\frac{\mathrm{d}\theta}{\mathrm{d}\tau}+q+2G(\xi_f)=2\theta \qquad (11)$$

where the function $G(\xi_f)$ is defined as

$$G(\xi_f)=\frac{1}{2\sqrt{\xi_f}}\int_0^{\xi_f}Y^n\mathrm{d}\xi \qquad (12a)$$

and

$$\xi_f=\left|\frac{\delta}{\Gamma}\exp\left|\frac{\Gamma\theta}{1+\beta_1\theta}\right|\right|^2$$

The asymptotic behavior of $G(\xi_f)$ can be shown to be

$$G(\xi_f) \sim \frac{\sqrt{\xi_f}}{2} - \frac{0.73}{3} n\xi_f \quad \text{for } \xi_f \to 0 \tag{12b}$$

and

$$G(\xi_f) \sim 1 - 2.10/\sqrt{\xi_f} \quad \text{for } \xi \gg 1 \text{ and } n=1 \tag{12c}$$

In Fig. 1, G is plotted as a function of ξ_f obtained from the numerical solution of Eqs. (10) and (12).

In this transient ignition process, there are three different characteristic stages. First, there is an inert heating stage in which the chemical reaction is frozen ($\xi_f = 0$). It is followed by a transition stage in which the process changes from kinetic to diffusion control. Finally, there is a heating stage where the chemical reaction is completely controlled by diffusion ($\xi_f \to \infty$), finally reaching a vigorous steady-state burning condition. In the following subsections, these three characteristic stages are analyzed.

Inert Stage

Solution of the inert stage problem ($G=0$) from Eq. (11) is given by

$$\phi = ½[1 - \exp(-2\tau)] \tag{13}$$

where $\phi = \theta/q$. We define a critical value of $\theta(\theta_c)$ in the following way:

$$\theta_c = \left| \frac{\Gamma}{\ell n(\Gamma/\delta)} - 1 \right|^{-1}$$

This critical value makes $\xi_f = 1$. In a first approximation, the nondimensional time needed to reach θ_c is given by

$$\tau_c = -\frac{1}{2} \ell n \left| 1 - \frac{2\theta_c}{q} \right| \tag{14}$$

Transition Stage

The inert stage ends when the nondimensional plate temperature approaches θ_c. When $\theta - \theta_c$ is of order $1/\Gamma_c$, the chemical reaction is important and has to be taken into account. Here, Γ_c means the appropriate Zel'dovich number and is defined as

$$\Gamma_c = \frac{\Gamma}{(1+\beta_1\theta_c)^2}$$

To study the transition stage, it is then convenient to introduce ψ of order unity defined as

$$\psi = 2\Gamma_c(\theta - \theta_c) \tag{15}$$

Thus, Eq. (11) transforms to

$$\frac{d\psi}{d\sigma} = K + 2G(e^{\psi}) \tag{16}$$

whre σ is the stretched nondimensional time defined as

$$\sigma = 2\Gamma_c(\tau - \tau_c) \tag{17}$$

and K is the nondimensional heating parameter $K = q - 2\theta_c$. Matching the inert stage solution is automatically satisfied with $\psi \sim K\sigma$ for $\sigma \to -\infty$.

Equation (16) can be numerically integrated, giving the evolution of ψ with σ for this transition stage. In Fig. 2, ψ is plotted as a function of σ for different values of the heating parameter K of order unity. At the end of this transition state, Eq. (16) reduces to

$$\frac{d\psi}{d\sigma} = K + 2 \tag{18}$$

Equation (18) shows that this ignition process cannot be characterized by a thermal runaway, due to the fact that, at the end of this transition stage, the reactant is rapidly consumed on the surface and the reaction is then controlled by the diffusion through the boundary layer, $K \ll 1$.

For values of $K \ll 1$, the transition stage is very large. Thus, a different approach must be employed to study the ignition process. In this case, the transition stage must match a weak asymptotic inert stage. Therefore, it is convenient to introduce a pretransition stage that couples the inert and transition stages. In this pretransition stage, $\xi_f \ll 1$, but differs from zero. Therefore as given by Eq. (12b),

$$G(\xi_f) \sim \frac{\sqrt{\xi_f}}{2} - \frac{0.73}{3} n\xi_f \quad \text{for } \xi_f \to 0$$

Equation (11) then transforms to

$$-\frac{d\theta}{d\tau} + q + \sqrt{\xi_f} - \frac{1.46}{3} n\xi_f = 2\theta \quad \text{for } \xi_f \to 0 \tag{19}$$

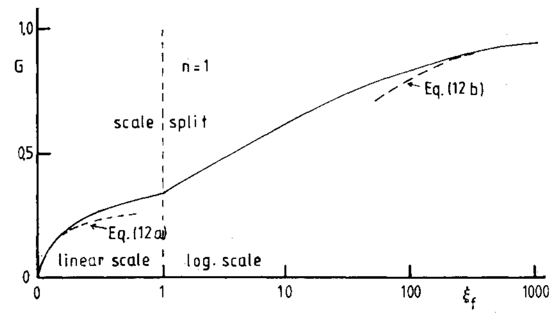

Fig. 1 Reaction parameter G as a function of ξ_f for $\eta = 1$.

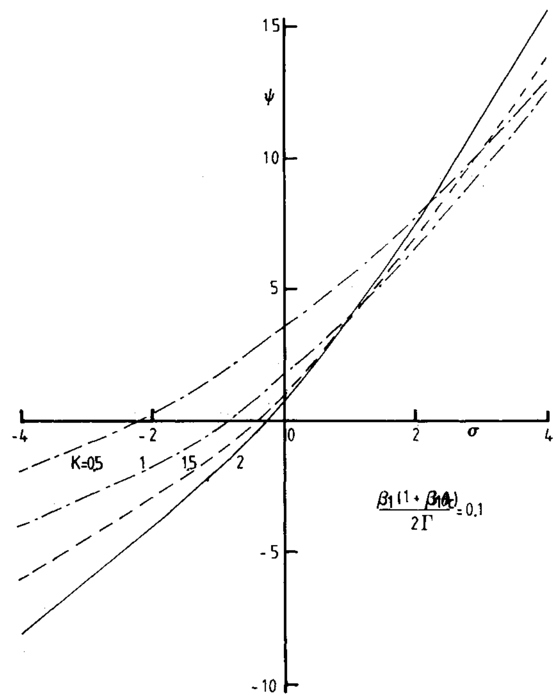

Fig. 2 Transition stage structure, showing ψ as a function of σ, for different values of the heating parameter K.

In the inert stage, the temperature of the plate will reach the value of $q/2$ asympotically for large times. Therefore, it is convenient to introduce the following expansions:

$$\theta=\theta_a+\epsilon\psi \text{ and } q=q_a+\epsilon q' \tag{20}$$

where $q_a=2\theta_a$ and the value of θ_a will be found later. Here ϵ is a small number and represents the inverse of the Zel'dovich number appropriate in this case and is defined as

$$\epsilon=1/\Gamma_a=(1+\beta_1\theta_a)^2/\Gamma \tag{21}$$

Introducing Eqs. (20) into Eq. (19), we obtain

$$-\frac{d\psi}{d\tau}+q+\Delta e^{\psi}\left|1-\frac{\beta_1\epsilon\psi^2}{(1+\beta_1\theta_a)^2}\right|-0.486n\Delta^2e^{2\psi}=2\psi \tag{22}$$

where

$$\Delta=\frac{\delta\Gamma_a}{\Gamma}\exp\left|\frac{\Gamma\theta_a}{1+\beta_1\theta_a}\right|$$

As shown by this relation, Δ depends on θ_a. Thus, the best way to define θ_a is to make $\Delta=1$. Matching with the inert stage indicates that

$$\psi\sim-\exp(-2\sigma)+q'/2 \text{ for } \sigma\to-\infty \tag{23}$$

where $\sigma=\tau-½\ell n\,|q_a\theta_a/2|$.

The asymptotic behavior given by Eq. (23) is to be used for the integration of Eq. (22). Close to the critical conditions denoted by the subscripts c, the solution to Eqs. (22) and (23) can be obtained through the introduction of the following expressions:

$$\psi=\psi_c+\sum_{n=0}^{\infty}\epsilon^{n/2}\psi_n$$

and

$$q'=q'_c+\sum_{n=0}^{\infty}\epsilon^n q'_n \tag{24}$$

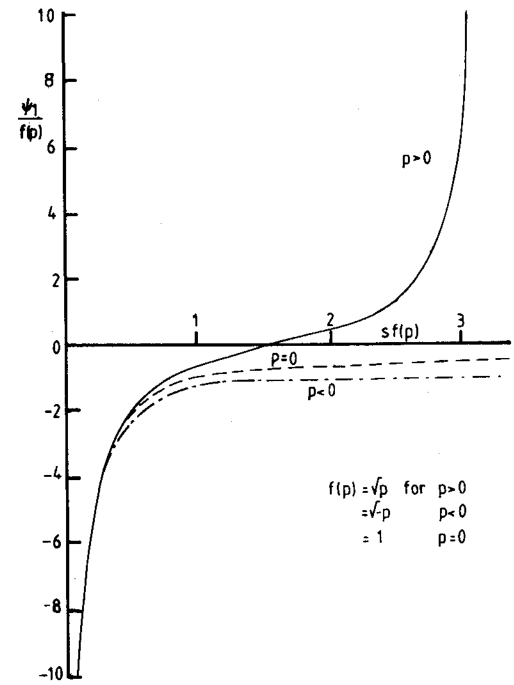

Fig. 3 Pretransition stage structure for different values of the parameter p (for values of $p>0$, ignition is obtained).

Defining the new time scale as $s=\sqrt{\epsilon}\,\sigma$, we obtain the following set of equations, resulting from the collection of the terms with the same power of ϵ:

$$e^{\psi_c}-2\psi_c=-q'_c \qquad \text{of order } \epsilon^{\circ} \tag{25}$$

$$(e^{\psi_c}-2)\psi_1=0 \qquad \text{of order } \epsilon^{½} \tag{26}$$

$$(e^{\psi_c}-2)\psi_2=\frac{d\psi_1}{ds}-e^{\psi_c}\frac{\psi_1^2}{2}-p \qquad \text{of order } \epsilon^1,\text{ etc.} \tag{27}$$

where

$$p=q'_1-\frac{e^{\psi_c}\beta_1\psi_c^2}{(1+\beta_1\theta_a)^2}-0.486ne^{2\psi_c}$$

Equation (25) gives the critical values of

$$\psi_c=\ell n2 \text{ and } q'_c=2\,|\ell n2-1| \tag{28}$$

Equation (26) is trivial. Equation (27) gives the evolution of ψ with time and thus transforms to

$$\frac{d\psi_1}{ds}=\psi_1^2+p \tag{29}$$

with the initial condition given by

$$\psi_1\to-\infty \text{ as } s\to0 \tag{30}$$

The solution to Eqs. (29) and (30) is given by

$$s=\frac{\pi}{2\sqrt{p}}+\frac{1}{\sqrt{p}}\text{arctg}\left(\frac{\psi_1}{\sqrt{p}}\right), \qquad \text{for } p>0 \tag{31}$$

$$s=\frac{1}{2\sqrt{-p}}\ell n\left|\frac{\psi_1-\sqrt{-p}}{\psi_1+\sqrt{-p}}\right|, \qquad \text{for } p<0 \tag{32}$$

$$s=-1/\psi_1, \qquad \text{for } p=0 \tag{33}$$

Figure 3 shows the ψ_1 profiles as a function of s for different values of p. Ignition occurs only for positive values of p. Therefore, this gives the correction to the critical value of the external heat flux to obtain ignition. The time needed to reach the apparent thermal runaway is given by

$$s=\pi/\sqrt{p}$$

or

$$\tau=\tau_c=\pi\left|\sqrt{\frac{\Gamma_a}{p}}+\frac{1}{2}\ell n\left|\frac{q_a\Gamma_a}{2}\right|\right| \quad \text{for } p>0 \tag{34}$$

with

$$p=q'_1-\frac{2(\ell n2)\beta_1}{(1+\beta_1\theta_a)}-1.944n$$

For negative values of p, that is when the critical external heat flux is less than that needed for ignition, ψ_1 reaches the steady-state value of

$$\psi_1=-\sqrt{-p} \text{ for } \tau\to\infty$$

In the time scale s, we obtain a thermal runaway at the end of the pretransition stage. However, a very short transition stage must follow at the end, which matches the final or diffusion-controlled stage.

Final Stage

In this final or diffusion-controlled stage, the equation

$$\frac{d\theta}{d\tau}=q-2\theta+2 \tag{35}$$

has to be solved with the initial conditions $\theta=\theta_c$ at $\tau=\tau_c$. The solution is

$$\theta = \tfrac{1}{2}(q+2) - \tfrac{1}{2}(K+2)\exp[-2(\tau-\tau_c)] \tag{36}$$

which shows that equilibrium $\theta=\frac{1}{2}(q+2)$ is reached at $\tau\to\infty$. Figure 4 schematically diagrams the evolution of the nondimensional temperature of the plate for different values of the parameter K.

Adiabatic Plate ($\alpha=0$)

In this particular case, the governing equations transform to

$$-\frac{\delta}{\Gamma}Y^n\exp\left(\frac{\Gamma\theta}{1+\beta_1\theta}\right) = \frac{1}{\sqrt{\chi}}\int_1^Y K(\bar{\chi},\chi)\,\mathrm{d}\bar{Y} \tag{37}$$

and

$$-\frac{\partial\theta}{\partial\tau}+q+\frac{\delta}{\Gamma}Y^n\exp\left(\frac{\Gamma\theta}{1+\beta_1\theta}\right) = \frac{1}{\sqrt{\chi}}\int_0^\theta K(\bar{\chi},\chi)\,\mathrm{d}\bar{\theta} \tag{38}$$

associated with a leading-edge value of $\theta=0$. In contrast with the previous limit of perfect conducting plate, we can identify only two stages in this limiting case of an adiabatic plate. There is an inert stage in which the chemical reaction can be neglected. It follows a transition stage that ends with a thermal runaway at a specific ignition time.

Inert Stage

In this inert stage, Eq. (38) transforms to

$$\frac{\partial\phi}{\partial\tau} = 1 - \frac{1}{\sqrt{\chi}}\int_0^\phi K(\bar{\chi},\chi)\,\mathrm{d}\bar{\phi} \tag{39}$$

where $\phi=\theta/q$. As shown by Treviño and Liñán,[11] Eq. (39) admits a self-similar solution of the form

$$\phi=\tau\Omega(\eta) \quad \text{with} \quad \eta=\chi/\tau^2 \tag{40}$$

Thus, Eq. (39) reduces to

$$\Omega - 2\eta\frac{\mathrm{d}\Omega}{\mathrm{d}\eta} = 1 - \frac{1}{\sqrt{\eta}}\int_0^\Omega K(\bar{\eta},\eta)\,\mathrm{d}\bar{\Omega} \tag{41}$$

The initial condition is given by $\Omega(\infty)=1$. In fact, it can be shown that the asympotic behavior of Ω for large values of η are given by

$$\Omega \sim 1-(1/2\sqrt{\eta}) \quad \text{for} \quad \eta\to\infty \tag{42}$$

On the other hand, the equilibrium condition or steady-state solution can be found as[11]

$$\Omega \sim \frac{\sqrt{3}\Gamma(4/3)\Gamma(1/3)}{2\pi\Gamma(5/3)}\sqrt{\eta} \quad \text{for} \quad \eta\to 0 \tag{43}$$

where $\Gamma(\ )$ corresponds to the gamma function.

In Fig. 5, Ω is shown as a function of η after numerical integration of Eq. (41).

Transition Stage

The chemical reaction is important only in regions close to the trailing edge, where the temperature reaches the highest value. In this reactive zone, the frozen nondimensional temperature ϕ can be expanded as

$$\phi=\phi_t(\tau_c)+\gamma(\chi-1)+\lambda(\tau-\tau_c)+\ldots \tag{44}$$

with $\phi_t(\tau_c)$ as the value of ϕ at the trailing edge at a critical time τ_c to be defined later. Also,

$$\gamma=\left(\frac{\partial\phi}{\partial\chi}\right) \quad \text{at } \chi=1 \text{ and } \tau=\tau_c$$

$$\lambda=\left(\frac{\partial\phi}{\partial\tau}\right) \quad \text{at } \chi=1 \text{ and } \tau=\tau_c$$

The nondimensional temperature of the plate θ can be assumed to be given by

$$\theta=\theta_c+(q\phi-\theta_c)+(\psi/\Gamma_e)+\mathcal{O}(1/\Gamma_e)^2 \tag{45}$$

where ψ is of order unity and θ_c the corresponding temperature at the critical time τ_c, which will be given later. Γ_e is the equilvalent Zeldovich number given by

$$\Gamma_e=\Gamma/(1+\beta_1\theta_c)^2 \tag{46}$$

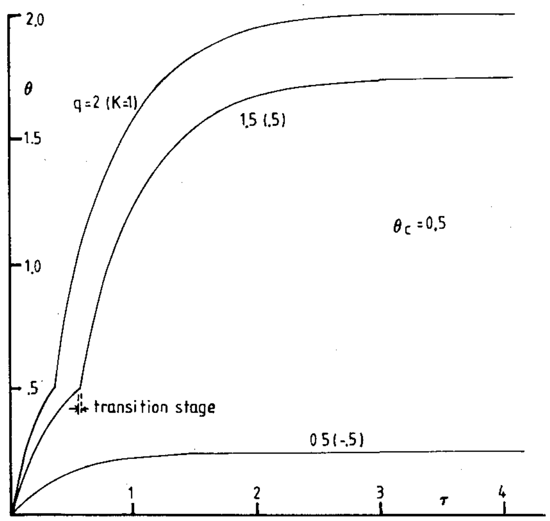

Fig. 4 Evolution of the nondimensional temperature of the plate with time for different values of the heating parameter K.

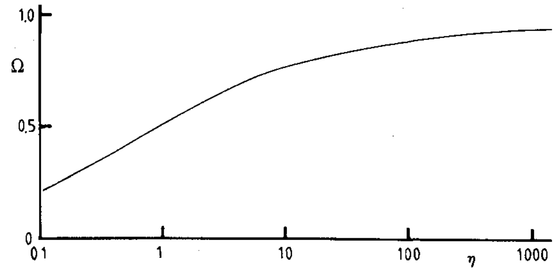

Fig. 5 Similarity parameter Ω as a function of η for an adiabatic plate.

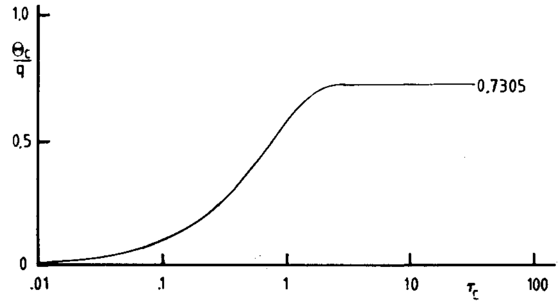

Fig. 6 Parameter θ_c/q as a function of the critical nondimensional time τ_c.

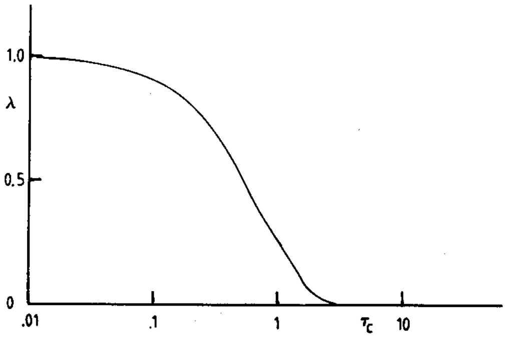

Fig. 7 Parameter λ, denoting the rage of change of the nondimensional temperature at the trailing edge for frozen flow, as a function of the critical nondimensional time τ_c.

Using Eq. (45), Eq. (38) transforms to

$$-\frac{1}{\Gamma_e}\frac{\partial\psi}{\partial t}+\Delta_c Y^n \exp\{q\Gamma_e[(\chi-1)+\lambda(\tau-\tau_c)]\}\exp\psi$$

$$=\frac{-1}{\Gamma_e\sqrt{\chi}}\int_0^{\psi} K(\bar{\chi},\chi)\mathrm{d}\bar{\psi} \tag{47}$$

where

$$\Delta_c=\frac{\delta}{\Gamma}\exp\left(\frac{\Gamma\theta_c}{1+\beta_1\theta_c}\right) \tag{48}$$

and is of order unity. Equation (47) dictates the following inner independent variables:

$$\zeta=\Gamma_e(1-\chi) \text{ and } \sigma=\Gamma_e(\tau-\tau_c) \tag{49}$$

Thus, Eq. (47) transforms to

$$-\frac{\partial\psi}{\partial\sigma}+\Delta_c Y^n \exp(q\sigma\lambda-q\zeta\gamma)\exp\psi=0 \tag{50}$$

which shows that this inner zone is transient reactive in nature. Therefore, the ignition characteristics are obtained at the trailing edge ($\zeta=0$),

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}\sigma}=\Delta_c Y_t^n \exp|q\sigma\lambda|\exp\psi_t \tag{51}$$

This equation has to be integrated with the condition

$$\psi\to 0 \text{ as } \sigma\to-\infty \tag{52}$$

which matches with the inert stage. Introducing the inner variables into Eq. (37) shows that, in a first approximation $Y_t=1$. Integration of Eq. (51) with Eq. (52) gives

$$\exp\psi_t=\left|1-\frac{\Delta_c}{q\lambda}\exp(q\sigma\lambda)\right|^{-1} \tag{53}$$

which shows that ignition ($\psi_t\to\infty$) occurs for

$$\Delta_c=q\lambda\exp(-q\sigma\lambda) \tag{54}$$

Equation (54) shows that Δ_c can be selected in such a way that runaway takes place at $\sigma=0$ ($\tau=\tau_c$). However, we follow a second procedure, fixing the value of $\Delta_c=1$ and obtaining the ignition time σ_I as

$$\sigma_I=\frac{\ln(q\lambda)}{q\lambda} \tag{55}$$

or

$$\tau_I=\tau_c+\frac{1}{\Gamma_e}\frac{\ln(q\lambda)}{q\lambda} \tag{56}$$

From the definition of Δ_c in Eq. (48), we obtain for θ_c,

$$\theta_c=\frac{\ln(\Gamma/\delta)}{\Gamma-\beta_1\ln(\Gamma/\delta)} \tag{57}$$

Once θ_c is known, τ_c can be obtained with the utilization of the similarity relations [Eq. (40)] as

$$\theta_c=\tau_c\Omega(1/\tau_c^2) \tag{58}$$

This relation is shown in Fig. 6. The parameter λ is then obtained as a function of τ_c as

$$\lambda=\Omega-2\eta\frac{\mathrm{d}\Omega}{\mathrm{d}\eta} \text{ at } \eta=\frac{1}{\tau_c^2} \tag{59}$$

This relation is shown in Fig. 7. The ignition time can be determined with the aid of Eq. (56).

Conclusions

In this work, a transient analysis was made of the catalytic ignition in a flat-plate, boundary-layer flow with an external energy flux. The plate was assumed to have finite thickness and thermal conductivity and an initial temperature equal to that of the freestream. Two types of analyses were made depending on the parameter α, which denotes the ratio between the thermal resistance of the gas and that of the plate. For high thermal conductivity in the plate (i.e., $\alpha\to\infty$), the temperature of the plate depends only on time. There are three characteristic stages on the transient process. First, there is an inert stage where the surface reaction can be neglected. It is followed by a transition stage in which the chemical reaction becomes important. In this transition stage, the reaction front translates to regions close to the leading edge, thus allowing the reaction to be controlled by diffusion. In the final diffusion-controlled stage, the temperature of the plate will increase until the steady-state equilibrium temperature is reached. On the other hand, for the adiabatic plate, there are only two characteristic stages. In the inert stage, the temperature of the plate increases faster at the trailing edge of the plate. Thus, only in regions close to the trailing edge is the chemical reaction important. At the end of the transition stage, there is a thermal runaway that instantaneously reaches the equilibrium temperature in a portion of the plate downstream of this position. Analytical expressions for the critical ignition time have been obtained for those two limiting cases.

From the definition of α [Eq. (7)], we have

$$\alpha=K\frac{\lambda p}{\sqrt{u_\infty L}}\frac{d}{L}$$

where $K=Pr^{2/3}/0.332\sqrt{\nu\rho}C_p\sim 1\ \mathrm{m^2\,K\sqrt{s}/J}$ for air.

For the typical values of $d/L\sim 0.1$ and $u_\infty L\sim 1\ \mathrm{m^2/s}$, we obtain finally

$$\alpha\sim\lambda p/10$$

or $\alpha\sim 10$ for metals and $\alpha\sim 0.1$ for ceramic materials. Therefore, the limit $\alpha\to\infty$ can be used for metals and the limit $\alpha\to 0$ for materials with very low thermal conductivities ($\lambda p\sim 1$ W/m·K).

Appendix

In this Appendix, we describe the numerical schemes used for the integration of the Eqs. (10) and (12). It is convenient to introduce a new coordinate $z=\xi^{1/4}$. Therefore, Eq. (10) takes the form

$$-Y^n=\frac{1}{z^{1/3}}\int_0^z\frac{\mathrm{d}\bar{Y}}{\mathrm{d}\bar{z}}\frac{\mathrm{d}\bar{z}}{[1-(\bar{z}/z)]^{1/3}} \tag{A1}$$

For small values of z, that is $z\to 0$, Y can be expressed as

$$Y\sim 1-az^b, \quad \text{for } z\to 0 \tag{A2}$$

Introducing Eq. (A2) into Eq. (A1), we can obtain the following values for constants a and b:

$$a=\frac{1}{2}\frac{\Gamma(1/3)}{[\Gamma(2/3)]^2}=0.7305$$

$$b=2/3$$

This relation will be employed in the numerical calculation. We divide the z coordinate in equal-spaced increments Δz. Thus, the position of the N point is given by $N\Delta z$. Equation (A1) transforms to

$$-Y^n z^{1/3}=\sum_{k=1}^{N}\int_{(K-1)\Delta z}^{k\Delta z}\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{z}}\frac{\mathrm{d}\bar{z}}{(z-\bar{z})^{1/3}} \tag{A3}$$

The derivative $\mathrm{d}Y/\mathrm{d}z$ can be approximated by a first-order Taylor expansion as

$$\frac{\mathrm{d}Y}{\mathrm{d}z}=\left.\frac{\mathrm{d}Y}{\mathrm{d}z}\right)_{k-1}+\left.\frac{\mathrm{d}^2Y}{\mathrm{d}z^2}\right)_{k-1}(z-z_{k-1})+\mathcal{O}(\Delta z^2) \tag{A4}$$

where the subindex $k-1$ means that the derivatives are to be evaluated at $z_{k-1}=(k-1)\Delta z$. We use central difference formulas given by

$$\left.\frac{\mathrm{d}Y}{\mathrm{d}z}\right)_{k-1}=\frac{Y_k-Y_{k-2}}{2\Delta z}$$

and

$$\left.\frac{\mathrm{d}^2Y}{\mathrm{d}z}\right)_{k-1}=\frac{Y_k-2Y_{k-1}+Y_{k-2}}{\Delta z^2}$$

Equation (A3) then can be written as

$$-Y_N^n z^{1/3}=\sum_{k=1}^{N}\left[\left.\frac{\mathrm{d}Y}{\mathrm{d}z}\right)_{k-1}-\left.\frac{\mathrm{d}^2Y}{\mathrm{d}z}\right)_{k-2}\cdot z_{k-1}\right]$$
$$\times\int_{(k-1)\Delta z}^{k\Delta z}\frac{\mathrm{d}\bar{z}}{(z-\bar{z})^{1/3}}+\sum_{k=1}^{N}\left.\frac{\mathrm{d}^2Y}{\mathrm{d}z^2}\right)_{k-1}\int_{(k-1)\Delta z}^{k\Delta z}\frac{\bar{z}\mathrm{d}\bar{z}}{(z-\bar{z})^{1/3}} \tag{A5}$$

Evaluating the integrals and after ordering terms, we obtain

$$\frac{33}{20}Y_N+Y_N^n z^{1/3}\Delta z^{1/3}=F(N-1) \tag{A6}$$

where

$$F(N-1)=-\frac{3}{4}\sum_{k=1}^{N-1}(Y_k-Y_{k-1})[(N+1-k)^{2/3}$$
$$-(N-k)^{2/3}]-\frac{9}{10}\sum_{k=1}^{N-1}(Y_k-2Y_{k-1}+Y_{k-2})\Big[(N+1-k)^{5/3}$$
$$-(N-k)^{5/3}-\frac{5}{3}(N-k)^{2/3}\Big]-\frac{9}{5}Y_{N-1}+\frac{3}{20}Y_{N-2} \tag{A7}$$

Therefore, the value of Y at the point $z=N\Delta z(Y_N)$ can be obtained from the algebraic equation (A6). In general for a noninteger n, iterative methods can be used to evaluate Eq. (A6). From the analysis illustrated above, it is necessary to have the values of Y at the two adjacent points. However, for $N=$ or $k=1$, we have only one point $Y_0=1$. Therefore, we can use Eq. (A2) to evaluate Y_1. This is given by

$$Y_1=1-a\Delta z^{2/3} \tag{A8}$$

The same problem arises in Eq. (A7) when $k=1$. In this case, the integral from Eq. (A5) can be obtained with the aid of Eq. (A8). Thus, $F(N-1)$ can be given by

$$F(N-1)=-\frac{3}{4}\sum_{k=2}^{N-1}(Y_k-Y_{k-1})[(N+1-k)^{2/3}$$
$$-(N-k)^{2/3}]-\frac{9}{10}\sum_{k=2}^{N-1}(Y_k-2Y_{k-1}+Y_{k-2})$$
$$\times\left[(N+1-k)^{5/3}-(N-k)^{5/3}-\frac{5}{3}(N-k)^{2/3}\right]-\frac{9}{5}Y_{N-1}$$
$$+\frac{3}{20}Y_{N-2}+\frac{2}{3}az^{1/3}\Delta z^{1/3}B\left(\frac{1}{N}\right)\left(\frac{2}{3},\frac{2}{3}\right) \tag{A9}$$

where $B(1/N)(2/3, 2/3)$ denotes the incomplete beta function.

To evaluate Eq. (12) numerically we introduce in the same way $z=\xi f^{3/4}$. Thus, Eq. (12) is then given by

$$G=\frac{2}{3z^{2/3}}\int_0^z Y^n\bar{z}^{1/3}\mathrm{d}\bar{z} \tag{A10}$$

Equation (A10) can be rewritten as

$$G_N=\frac{2}{3z^{2/3}}\sum_{K=1}^{N}\int_{(k-1)\Delta z}^{k\Delta z}Y^n\bar{z}^{1/3}\mathrm{d}\bar{z} \tag{A11}$$

Here, Y^n can be expanded in Taylor series given by

$$Y^n=Y_{k-1}^n+nY_{k-1}^{n-1}\left.\frac{\mathrm{d}Y}{\mathrm{d}z}\right)_{k-1}(z-z_{k-1})+\mathcal{O}(\Delta z^2) \tag{A12}$$

where

$$\left.\frac{\mathrm{d}Y}{\mathrm{d}z}\right)_{k-1}=\frac{Y_k-Y_{k-2}}{2\Delta z}$$

Introducing Eq. (A12) into Eq. (A11), finally results in

$$G_N=\frac{2\Delta z^{4/3}}{3z^{2/3}}\left\{\frac{3}{4}\sum_{k=2}^{N}\left[Y_{k-1}^n-\frac{n}{2}Y_{k-1}^n(Y_k-Y_{k-2})(k-1)\right]\right.$$
$$\times[k^{4/3}-(k-1)^{4/3}]+\frac{3n}{14}\sum_{k=2}^{N}Y_{k-1}^n(Y_k-Y_{k-2})$$
$$\left.\times[k^{7/3}-(k-1)^{7/3}]+G_1\right\}$$

where $G_1=1-(an/2)\Delta z^{2/3}$ is obtained from the asymptotic solution for small values of z given by Eq. (12b).

Acknowledgment

The work of the second author was supported by the Fondo de Estudios e Investigaciones Ricardo J. Zevada of Mexico. Support was also provided by the Ministerio de Asuntos Exteriores of Spain, the Dirección General de Intercambio Académico of the National University of Mexico through the

Scientific and Cultural Exchange Program between Mexico and Spain.

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